Existence of Electron spin

The Dirac's Hamiltonian in a stationary central field represented by potential $V(r)$ is

$$H = c\vec{\alpha}\cdot\vec{P} + \beta mc^2 + V(r) \qquad \text{— (1)}$$

For a particle under the influence of a central force the torque is zero.

Therefore the angular momentum is a constant of

$$L = r \times P$$

motion in central field.

The operator commute with H is a constant of motion. Let us examine this for x-component of momentum.

$$i\hbar \frac{dL_x}{dt} = [L_x, H] = [L_x, c\vec{\alpha}\cdot\vec{P} + mc^2 + V(r)]$$

$$= [L_x, c\{\alpha_x P_x + \alpha_y P_y + \alpha_z P_z\} + \beta mc^2 + V(r)]$$

$$= [L_x, c\alpha_x P_x] + [L_x, c\alpha_y P_y] + [L_x, c\alpha_z P_z]$$

$$+ [L_x, \beta mc^2] + [L_x, V(r)] \qquad \text{— (2)}$$

But L_x commute with the term P_y, P_z

$$[L_x, c\alpha_x P_x] = [L_x, \beta mc^2] = [L_x, V(r)] = 0 \quad (2)$$

Therefore, we get

$$i\hbar \frac{dL_x}{dt} = [L_x, c\alpha_y P_y] + [L_x, c\alpha_x P_z] \quad -(3)$$

$$= [L_x, c\alpha_y P_y] = c\alpha_y [L_x, P_y]$$

$$= c\alpha_y [yP_z - zP_y, P_y]$$

$$= c\alpha_y \{[yP_z, P_y] - [zP_y, P_y]\}$$

$$= c\alpha_y [[y, P_y]P_z - y[P_z, P_y] - 0]$$

$$= c\alpha_y [i\hbar P_z - 0] = c\alpha_y i\hbar P_z \quad -(4)$$

We can also calculate

$$[L_x, c\alpha_z P_z] = c\alpha_z \cdot (-i\hbar P_y) \quad -(5)$$

$$i\hbar \frac{dL_x}{dt} = c\alpha_y (i\hbar P_z) + c\alpha_z(-i\hbar P_y)$$

$$= -i\hbar c(\alpha_z P_y - \alpha_y P_z)$$

$$\neq 0 \quad -(6)$$

$$\frac{dL_x}{dt} \neq 0, \quad L_x \neq \text{Constant}$$

Hence according to Dirac's theory the x- component (3)
of orbital angular momentum moving in a central electrostatic field is not a constant of motion. So we can find another operator such that the commutator of its x-component with H is equal and opposite of the right side of 5. So that the some of this operator and L is then a constant of motion can be interpreted as the total angular momentum.

It is not difficult of see that the desire operator is multiple of $\vec{\sigma}'$

$$\vec{\sigma}' = \begin{bmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{bmatrix}$$

Let us take x-component of $\vec{\sigma}'$

$$i\hbar \frac{d\sigma x'}{dt} = [\sigma x', H]$$

$$\vec{\sigma}x' = \begin{bmatrix} \sigma x & 0 \\ 0 & \sigma x \end{bmatrix}$$

i.e $$i\hbar \frac{d\sigma x'}{dt} = [\sigma x', (c\vec{\alpha}\cdot p + \beta mc^2 + V(r))]$$

$$= [\sigma x', c(\alpha_x P_x + \alpha_y P_y + \alpha_z P_z) + \beta mc^2 + V(r)] \quad -(7)$$

But $\sigma x'$ commutes with every quantity in above except α_y and α_z

$$\sigma x'\beta - \beta\sigma x' = \begin{bmatrix} \sigma x & 0 \\ 0 & \sigma x \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \sigma x & 0 \\ 0 & \sigma x \end{bmatrix}$$

$$= \begin{bmatrix} \sigma x & 0 \\ 0 & -\sigma x \end{bmatrix} - \begin{bmatrix} \sigma x & 0 \\ 0 & -\sigma x \end{bmatrix} = 0$$

$$\sigma x'\alpha x - \alpha x\sigma x' = \begin{bmatrix} \sigma x & 0 \\ 0 & \sigma x \end{bmatrix}\begin{bmatrix} 0 & \sigma x \\ \sigma x & 0 \end{bmatrix} - \begin{bmatrix} 0 & \sigma x \\ \sigma x & 0 \end{bmatrix}\begin{bmatrix} \sigma x & 0 \\ 0 & \sigma x \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \sigma x^2 \\ \sigma x^2 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \sigma x^2 \\ \sigma x^2 & 0 \end{bmatrix} = 0$$

from equation (7)

$$i\hbar\frac{d\sigma x'}{dt} = [\sigma x', c\alpha_y P_y + c\alpha_z P_z] = [\sigma x', c\alpha_y P_z] + [\sigma x', c\alpha_z P_z]$$

(8)

$$[\sigma x', \alpha y] = \sigma x'\alpha y - \alpha y\sigma x'$$

$$= \begin{bmatrix} \sigma x & 0 \\ 0 & \sigma x \end{bmatrix}\begin{bmatrix} 0 & \sigma y \\ \sigma y & 0 \end{bmatrix} - \begin{bmatrix} 0 & \sigma y \\ \sigma y & 0 \end{bmatrix}\begin{bmatrix} \sigma x & 0 \\ 0 & \sigma x \end{bmatrix}$$

$$= \begin{bmatrix} 0 & i\sigma z \\ i\sigma z & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i\sigma z \\ -i\sigma z & 0 \end{bmatrix} = 2i\alpha_z$$

Similarly $[\sigma x', \sigma z] = -2i\alpha y$

from equation (8)

$$i\hbar\frac{d\sigma x'}{dt} = c\,2i\alpha_z P_y + c(-2i\alpha_y)P_z \quad — (9)$$

$$+ 2ic(\alpha_z P_y - \alpha_y P_z)$$

Multiplying both sides by $\frac{1}{2}\hbar$

$$i\hbar \frac{d}{dt}\left(\frac{1}{2}\hbar \sigma x'\right) = i\hbar c\,(\alpha_z P_z - \alpha_y P_z) \quad \text{--- (10)}$$

from equation 6 & 10

$$i\hbar \frac{dLx}{dt} + i\hbar \frac{d}{dt}\left(\frac{1}{2}\hbar \sigma x'\right) = 0$$

$$= \frac{d}{dt}\left(Lx + \frac{1}{2}\hbar \sigma x'\right) = 0$$

$$Lx + \frac{1}{2}\hbar \sigma x' = \text{constant} \quad \text{--- (11)}$$

Hence the quantity $J = \left[L + \frac{1}{2}\hbar \sigma'\right]$ commute with H can be taken as total angular momentum then $S = \frac{1}{2}\hbar \vec{\sigma}'$

as the spin angular momentum of a electron.

Hence Dirac's theory automatically endows the spin motion of electron.